Arithmetic-Geometric Mean and Logarithmic Mean

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Abstract

We present the Gauss' Arithmetic-Geometric Mean and study its relationship with the Logarithmic Mean. In particular we discuss a problem by Vamanamurthy and Vourian, [5].

1 Introduction

The Arithmetic-Geometric Mean (AGM) has been known for almost 300 years, but even in our time, some of the related problems remain unsolved. The AGM was discovered by Lagrange, and 7 to 9 years later by Gauss, who rediscovered it when he was a teenager.

It is well known that if x and y be positive numbers, then the Arithmetic Mean and the Geometric Mean are defined as follows:

$$A(x,y) = \frac{x+y}{2} \text{ and } G(x,y) = \sqrt{xy}$$

It is less known that these means may be combined to define the *Arithmetic-Geometric Mean* (AGM) [5]:

$$AGM(x,y) = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$$

where

$$x_0 = x, \quad y_0 = y$$

and

$$x_{n+1} = A(x_n, y_n), \quad y_{n+1} = G(x_n, y_n).$$

*This paper presents results obtained during the Faculty - Student Reasearch Project with Dr. Jakub Jasinski.

The reader is encouraged to follow Exercise 69, p.608 of [4] to prove that indeed the sequences x_n and y_n converge to the same limit. The truth is that x_n and y_n converge to the same number, say a, very rapidly. So if we could find x and y such that a is an interesting number, like π, AGM could be useful to compute that number very accurately.

The main purpose of our paper is to introduce the AGM and to give a partial solution to an open problem posed by M.K. Vamanurthy and M. Vuorinen [5] p. 165. In our paper we will use some definitions from [1], [5], and [2].

Now that the reader can not wait to hear more about the AGM let us begin.

As we mentioned earlier the sequences x_n and y_n converge to AGM very rapidly. Here is a little Maple program showing how quick is the convergence:

```
accuracy:=10:
x := 5 :
y := 2 :
counter:= 0 :
    while evalf(abs(x − y),accuracy+5) > 0.1^(accuracy) do
        am :=evalf((x + y)/2,accuracy+5):
        gm :=evalf(sqrt(x × y),accuracy+5):
        counter:=counter+1;  print(counter);
        x := am:
        y := gm:
    od :
```

And here is table of results:

accuracy (decimal places)	5	10	100	500	$1,000$	$10,000$
# of iterations	3	4	7	10	11	14

Observe that if $x > y$ then for all $n = 0, 1, 2, ...$ we have $x_n > AGM(x, y) > y_n$. So if we want the first 10,000 digits of $AGM(5, 2)$, we need only 14 iterations. Here is an algorithm that was discovered by Eugene Salamin and Richard Brent in 1976 [2] p. 688 to determine π with help of the of AGM:

Set $a_0 = 1$, $g_0 = \frac{1}{\sqrt{2}}$ and $s_0 = \frac{1}{2}$.

For $k = 1, 2, 3, ...$ compute

$$a_k = \frac{a_{k-1} + g_{k-1}}{2}$$
$$g_k = \sqrt{a_{k-1} \times g_{k-1}}$$
$$c_k = a_k^2 - g_k^2$$
$$s_k = s_{k-1} - 2^k c_k$$
$$p_k = \frac{2a_k^2}{s_k}$$

Then it can be proved, [3]that p_k converges quadratically to π i.e.,

number iterations	1	2	3	4	....	n
decimals of π	1	4	9	16		n^2

To continue we have to introduce a few more definitions:

Definition 1 *Logarithmic Mean (LM) is defined as,*

$$LM(x,y) = \frac{x-y}{\log x - \log y},\ \textit{while } x \neq y$$

and

$$LM(x,y) = x \textit{ when } x = y$$

Definition 2 *Sub "t" is used for modification of means as follows,*

$$M_t(x,y) = M\left(x^t, y^t\right)^{\frac{1}{t}}, \textit{where } t \in \mathbb{R},\ \textit{except } t = 0,$$

where $M = A$, G, AGM, *and* LM.

For example

$$AGM_t = AGM\left(x^t, y^t\right)^{\frac{1}{t}},$$

$$LM_t(x,y) = LM\left(x^t, y^t\right)^{\frac{1}{t}}.$$

2 AGM_t and LM are continuous

Since $LM(x,y)$ is a piecewise defined function and $AGM(x,y)$ is defined recursively we would like to take a moment to prove their continuity. This will support our conjecture below.

To prove that the AGM is continuous we may use the following lemma.

Lemma 3 *Let* $G \subseteq \mathbb{R}^2$ *be an open set, and let* $D = G \times [a,b]$. *If* $f : D \to \mathbb{R}$ *is a continuous then* $g(x,y) = \int_a^b f(x,y,z)dz$ *is continuous on* G.

Proof. Let $(x_0, y_0) \in G$ be an arbitrary point. To show that g is continuous at (x_0, y_0) let us pick an $\varepsilon > 0$. By the continuity of f for each $z_0 \in [a,b]$ there exists a δ such that if $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ then

$$|f(x_0,\ y_0, z_0) - f(x,y,z_0)| < \frac{\varepsilon}{b-a}. \tag{1}$$

Inequality 1 holds not only for z_0 but for all z from some open neighborhood of z_0. By the compactness of $[a,b]$ we conclude that there exists one δ_0 such that

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_0 \Rightarrow |f(x_0,\ y_0, z) - f(x,y,z)| < \frac{\varepsilon}{b-a}$$

for all $z \in [a,b]$ and $(x,y) \in G$. Now assume that $(x,y) \in G$ is such that $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_0$.

We have:

$$\begin{aligned}
|g(x_0,\, y_0) - g(x,y)| &= \left| \int_a^b f(x_0, y_0, z)dz - \int_a^b f(x,y,z)dz \right| \\
&= \left| \int_a^b (f(x_0,y_0,z) - f(x,y,z))dz \right| \\
&\le \int_a^b |f(x_0,y_0,z) - f(x,y,z)|dz \\
&\le \int_a^b \frac{\varepsilon}{b-a}dz \\
&= \left[\frac{\varepsilon}{b-a} \right]_a^b \\
&= \frac{\varepsilon}{b-a}b - \frac{\varepsilon}{b-a}a = \varepsilon.
\end{aligned}$$

This concludes the proof of our lemma. ■

Theorem 4 *$AGM(x,y)$ is a continuous function on $[0,\infty)^2$.*

Proof. Let $f(x,y,z) = \dfrac{1}{\sqrt{x^2\cos^2 z + y^2 \sin^2 z}}$. f is a continuous function on $\mathbb{R}^2 \setminus \{(0,0)\} \times \mathbb{R}$. Gauss proved [2] p.484 that for $x, y \ge 0$

$$AGM(x,y) = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} f(x,y,z)dz$$

hence it follows from the lemma that the $AGM(x,y)$ is continuous at $(x,y) \neq (0,0)$.

To proof that the $AGM(x,y)$ is continuous at $(0,0)$ let us pick $\varepsilon > 0$ and let $\delta = \varepsilon$. Assume that $\sqrt{x^2+y^2} < \delta$. We have

$$\begin{aligned}
|AGM(x,y) - AGM(0,0)| &= |AGM(x,y)| \\
&< \max\{|x|\,,\ |y|\} \\
&= \max\{\sqrt{x^2}, \sqrt{y^2}\} \\
&\le \sqrt{x^2+y^2} = \varepsilon
\end{aligned}$$

That concludes that the AGM is continuous function for any $x, y \ge 0$. ■

Theorem 5 *LM is a continuous function on $(0,\infty)^2$.*

Proof. LM is piecewise defined so we shall have two cases:
Case *1. When the point (x_0, y_0) is on the diagonal $\{(x,x) : x > 0\}$.*

We must show that for every $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that whenever $x, y > 0$

$$\sqrt{(x-x_0)^2+(y-x_0)^2} < \delta \implies |L(x,y) - L(x_0,x_0)| < \varepsilon$$

If $x = y$ then the above implication holds with $\delta = \varepsilon$. If $x \neq y$ then by the mean value theorem [4]

$$|LM(x,y) - LM(x_0,x_0)| = \left|\frac{x-y}{\log x - \log y} - x_0\right| = \left|\frac{x-y}{\frac{1}{c}(x-y)} - x_0\right| = |c - x_0| \tag{2}$$

where c is between x and y.

$$|c - x_0| = |c - x + x - x_0| \leq |c - x| + |x - x_0| \tag{3}$$

since

$$|x - x_0|, |y - x_0| < \delta \text{ and } |c - x| < |x - y|$$

we obtain

$$|c-x| + |x-x_0| \leq |x-y| + \delta \leq |x-x_0| + |x_0 - y| + \delta \leq 3\delta$$

Therefore

$$|c - x_0| \leq 3\delta.$$

Hence by 2 and 3 it is clear that if we take $\delta < \frac{\varepsilon}{3}$ the implication

$$\sqrt{(x-x_0)^2+(y-x_0)^2} < \delta \implies |L(x,y) - L(x_0,x_0)| < \varepsilon$$

holds for all $x, y > 0$.

Case *2. When point is not on the diagonal* then $LM(x,y)$ is continuous as a composition of continuous functions such as $\frac{x}{y}$, $x-y$, and $\log(x)$.

That concludes the proof that the Logarithmic Mean is continuous. ■

The function x^t is continuous for $x > 0$ so composed with Theorems 4 and 5 we obtain the following result:

Corollary 6 *If $t > 0$ then $AGM_t(x,y)$ and $LM_t(x,y)$ are continuous functions on $(0,\infty)^2$.*

3 $AGM_t \geq LM$ for some $t \in (0,1)$?

As we mentioned at the beginning, we have a partial proof for an open problem that we found in the article [5]:

Problem 7 *Is it true that $AG_t(x,y) \geq L(x,y)$ for some $t \in (0,1)$ where $x, y \in \mathbb{R}^+$?*

Theorem 8 *For positive x and y both $AGM_t(x,y)$ and $LM_t(x,y)$ are continuous, strickly increasing functions of t.*

Proof. See Theorem 1.2 of [5]. ■

Theorem 9 $AG_t(x,y) < L(x,y)$ *for all* $t \in \left(0, \frac{2}{3}\right)$ *and* $x, y \in \mathbb{R}^+$.

Proof. From [5] Theorem 3.6 we know that the inequality $AG(x,y) \leq L_{\frac{3}{2}}(x,y)$ holds for all $x, y > 0$. From our Definition 2 $L_{\frac{3}{2}}(x,y) = L\left(x^{\frac{3}{2}}, y^{\frac{3}{2}}\right)^{\frac{2}{3}}$, so we obtain $AG(x,y) \leq L\left(x^{\frac{3}{2}}, y^{\frac{3}{2}}\right)^{\frac{2}{3}}$ and $AG(x,y)^{\frac{3}{2}} \leq L\left(x^{\frac{3}{2}}, y^{\frac{3}{2}}\right)$. Now, let us substitute $p = x^{\frac{3}{2}}$ and $q = y^{\frac{3}{2}}$, then $AG\left(p^{\frac{2}{3}}, q^{\frac{2}{3}}\right)^{\frac{3}{2}} = AG_{\frac{2}{3}}(p,q) \leq L(p,q)$. By Theorem 8 $AG_t(x,y) < L(x,y)$ holds for all $t \in \left(0, \frac{2}{3}\right)$. ■

Therefore we can exclude any values of t between 0 and $\frac{2}{3}$, since our AGM_t and LM_t are strictly increasing functions of $t \in (0,1)$.

For the rest of a problem i.e., where t $\in \left[\frac{2}{3}, 1\right)$ we have the following conjecture.

Conjecture 10 *For every number* $t \in \left[\frac{2}{3}, 1\right)$ *there exists a constant* $m > 0$ *such that if* $y > mx > 0$ *then* $AGM_t(x,y) < L(x,y)$.

Our conjecture is based on computer calculations. To illustrate our results we present "matrices" where x and y play the role of the indecies and if $AG_t(x,y) \geq L(x,y)$ then we have a white dot, otherwise we have a black dot. Now we are ready to take a look at the matrices for $t = 0.6667$, 0.667, 0.67, and 0.7, Figures 1-4.

As we can see from these graphs, that the slope m of the line separating the vertical black region from the white area is rising rapidly as we increase t. We almost can not see the black area when $t = 0.70$ because the slope $m = 87$. We created another Maple program that would calculate these slopes and here are the results:

t	slope m
0.68	13
0.7	87
0.75	1.16×10^4
0.8	1.40×10^8
0.85	7.10×10^{20}
0.9	1.10×10^{107}
0.95	3.4×10^{10210}

As we can see from this table, the slope m increases, and increases very rapidly. Since both functions that we are working with are continuous functions of $t, x,$ and y and slope is rising we conjecture that there is no such $t \in (0,1)$ that will satisfy $AG_t(x,y) \geq L(x,y)$ for all $x, y \in (0, \infty)$.

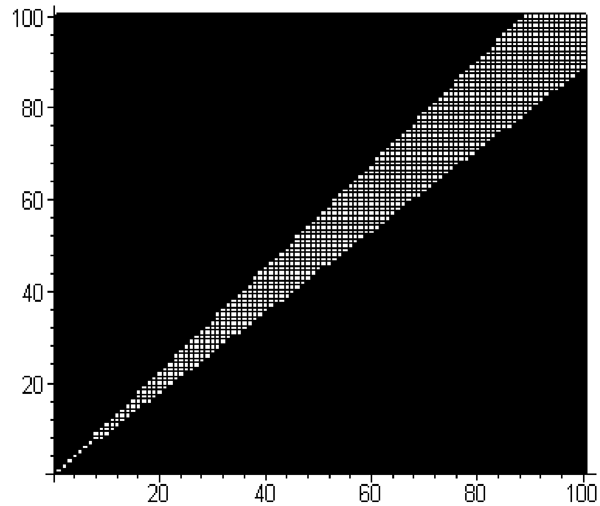

Figure 1: t=0.6667

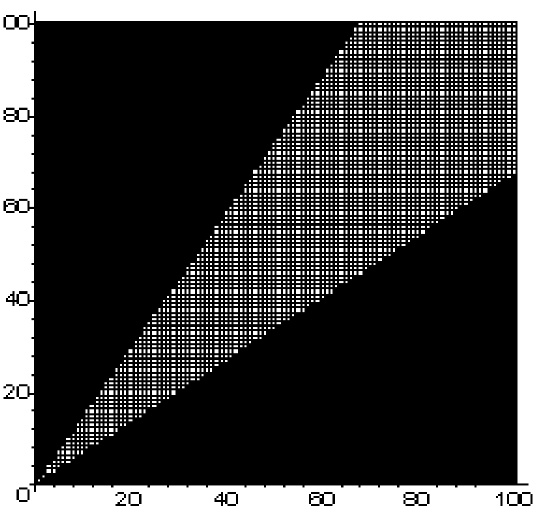

Figure 3: t=0.667

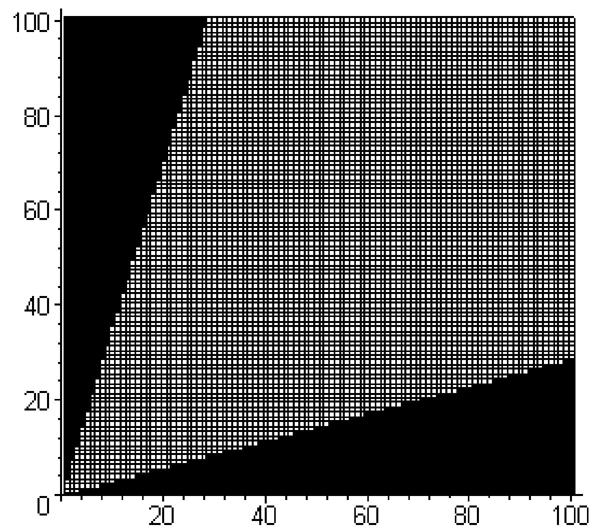

Figure 4: t=0.67

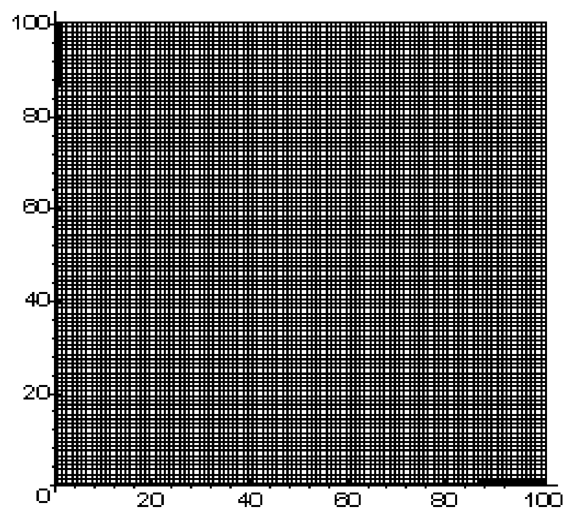

Figure 5: t=0.7

4 Acknowledgment

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References

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